Journal of Computational and Applied Mathematics 224 (2009) 127–132

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A class of logarithmically completely monotonic functions associated with the gamma function

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ARTICLE INFO

Article history:
Received 16 February 2008
Received in revised form 8 April 2008

MSC:
primary 26A48
26D07
secondary 33B15

Keywords:
Logarithmically completely monotonic function
Completely monotonic function
Gamma function
Necessary condition
Sufficient condition
Sufficient and necessary condition

ABSTRACT

In this article, a necessary condition, several sufficient conditions and a sufficient and necessary condition for a class of functions involving the gamma function to be logarithmically completely monotonic are established, and then some known results are extended.

1. Introduction

Recall [8, Chapter XIII] and [16, Chapter IV] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^k f^{(k)}(x) \geq 0 \tag{1}$$

for $x \in I$ and $k \geq 0$. The set of all completely monotonic functions on I is denoted by $CM(I)$.

Recall also [1,12] that a positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^k [\ln f(x)]^{(k)} \geq 0 \tag{2}$$

for $k \in \mathbb{N}$ on I. The class of all logarithmically completely monotonic functions on I is denoted by $LCM(I)$.

For any given interval I, it was proved explicitly in [2,9,12,14] that

$$LCM(I) \subset CM(I), \tag{3}$$

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0377-0427/$ – see front matter
doi:10.1016/j.cam.2008.04.028

among other things. For more information on the class of logarithmically completely monotonic functions, please refer to [2,4,6,9,11,13] and related references therein.

It is well known that the classical Euler gamma function is defined and denoted for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \tag{4}$$

The logarithmic derivative of $\Gamma(x)$, denoted by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{5}$$

is called the psi function, and $\psi^{(k)}(x)$ for $k \in \mathbb{N}$ are called the polygamma functions.

For $a \geq 0$ and $b, c, r \in \mathbb{R}$ with $r \neq 0$, define

$$f_{a,b,c,r}(x) = \left[\frac{\sqrt[x]{\Gamma(x+1)}}{x^c}\left(1+\frac{a}{x}\right)^{x+b}\right]^r \tag{6}$$

in $x \in (0, \infty) \triangleq \mathbb{R}^+$.

In [12], it was proved that $f_{0,b,c,1} \in LCM(\mathbb{R}^+)$ if $c \geq 1$.

In [13], it was showed that $f_{1,0,1,1} \in LCM(\mathbb{R}^+)$.

In [5], necessary and sufficient conditions on c for $f_{1,0,c,1}(x)$ or its reciprocal to be logarithmically completely monotonic in $\mathbb{R}^+$ were presented: $f_{1,0,c,1}(x) \in LCM(\mathbb{R}^+)$ if and only if $c \geq 1$ and $f_{1,0,c,-1}(x) \in LCM(\mathbb{R}^+)$ if and only if $c \leq 0$.

In [15], a sufficient condition on b such that $f_{1,b,1,1}(x) \in LCM(\mathbb{R}^+)$ was obtained.

In [10], some more general results are established: If $2a \leq 3b \leq -3c$, then $f_{a,b,c,1}(x) \in LCM(\mathbb{R}^+)$; If $2a \leq 3b$ and $1 + 2b + c \geq 0$, then $f_{a,b,c,-1}(x) \in LCM(\mathbb{R}^+)$.

In this article, a necessary condition such that $f_{a,b,c,r} \in CM(\mathbb{R}^+)$ and several sufficient conditions and a sufficient and necessary condition such that $f_{a,b,c,r} \in LCM(\mathbb{R}^+)$ will be established, which extend some known results mentioned above.

Our main results are as follows.

Theorem 1. *If $f_{a,b,c,r} \in CM(\mathbb{R}^+)$, then $rc \geq \max\{r, -br\operatorname{sgn}(a)\}$.*

Theorem 2. *Let $r > 0$.*

(1) *If $b \leq a$ and $c \geq 1 + 2a - b$, then $f_{a,b,c,r} \in LCM(\mathbb{R}^+)$;*
(2) *If $b > a$ and $c \geq 1 + 2a - b + (b-a)\exp(\frac{a}{a-b})$, then $f_{a,b,c,r} \in LCM(\mathbb{R}^+)$.*

Theorem 3. *Let $r < 0$.*

(1) *If $b \leq 2a$ and $c \leq -2a$, then $f_{a,b,c,r} \in LCM(\mathbb{R}^+)$;*
(2) *If $b > 2a$ and $c \leq -b$, then $f_{a,b,c,r} \in LCM(\mathbb{R}^+)$.*

Theorem 4. *When $r < 0$, $a > 0$ and $b > 2a$, the sufficient and necessary condition for $f_{a,b,c,r} \in LCM(\mathbb{R}^+)$ is $c \leq -b$.*

2. Lemmas

In order to show our theorems, the following lemmas are necessary.

Lemma 1 (*[3,7]*). *For $n \in \mathbb{N}$ and $x \in \mathbb{R}^+$,*

$$\psi^{(n)}(x) = (-1)^{n+1}\int_0^\infty \frac{t^n}{1-e^{-t}} e^{-xt} dt, \tag{7}$$

$$\psi^{(n-1)}(x+1) = \psi^{(n-1)}(x) + \frac{(-1)^{n-1}(n-1)!}{x^n}, \tag{8}$$

$$\frac{1}{x^n} = \frac{1}{\Gamma(n)}\int_0^\infty t^{n-1} e^{-xt} dt. \tag{9}$$

Lemma 2 (*[7]*). *As $x \to \infty$,*

$$\ln\Gamma(x) = \left(x - \frac{1}{2}\right)\ln x - x + \frac{\ln(2\pi)}{2} + O\left(\frac{1}{x}\right), \tag{10}$$

$$\psi(x) = \ln x - \frac{1}{2x} + O\left(\frac{1}{x^2}\right). \tag{11}$$

Lemma 3. *Let*

$$\phi_{a,b,c,r}(x) = r\left[\psi''(x) + \frac{2}{x^3} + \frac{b+c}{x^2} - \frac{2}{x} + \frac{2a(a-b)}{(x+a)^3} + \frac{2a-b}{(x+a)^2} + \frac{2}{x+a}\right] \tag{12}$$

for $x \in \mathbb{R}^+$, *then*

$$\phi_{a,b,c,r}(x) = r\int_0^\infty \rho_{a,b,c}(t)te^{-xt}\mathrm{d}t, \tag{13}$$

where

$$\rho_{a,b,c}(t) = b + c - \frac{a(b-a)t + b - 2a}{e^{at}} - \frac{2(1-e^{-at})}{t} - \frac{t}{e^t - 1}. \tag{14}$$

Proof. Using formulas (7) and (9) in Lemma 1 gives

$$\begin{aligned}\phi_{a,b,c,r}(x) &= r\left[-\int_0^\infty \frac{t^2e^{-xt}}{1-e^{-t}}\mathrm{d}t + \int_0^\infty t^2e^{-xt}\mathrm{d}t + (b+c)\int_0^\infty te^{-xt}\mathrm{d}t\right.\\ &\quad\left. -2\int_0^\infty e^{-xt}\mathrm{d}t + a(a-b)\int_0^\infty t^2e^{-(x+a)t}\mathrm{d}t + (2a-b)\int_0^\infty te^{-(x+a)t}\mathrm{d}t + 2\int_0^\infty e^{-(x+a)t}\mathrm{d}t\right]\\ &= r\int_0^\infty\left[b+c-\frac{a(b-a)t+b-2a}{e^{at}} - \frac{2(1-e^{-at})}{t} - \frac{t}{e^t-1}\right]te^{-xt}\mathrm{d}t.\end{aligned}$$

Lemma 3 is proved. □

3. Proofs of the theorems

Now we are in a position to prove our theorems.

Proof of Theorem 1. Taking logarithm and differentiating directly yield

$$\ln f_{a,b,c,r}(x) = r\left[(x+b)\ln\left(1+\frac{a}{x}\right) + \frac{\ln\Gamma(x+1)}{x} - c\ln x\right] \tag{15}$$

and

$$[\ln f_{a,b,c,r}(x)]' = r\left[\ln\left(1+\frac{a}{x}\right) - \frac{a(x+b)}{x(x+a)} + \frac{x\psi(x+1) - \ln\Gamma(x+1)}{x^2} - \frac{c}{x}\right]. \tag{16}$$

If $f_{a,b,c,r} \in CM(\mathbb{R}^+)$, then

$$[f_{a,b,c,r}(x)]' \le 0, \quad x \in \mathbb{R}^+. \tag{17}$$

Hence

$$x[\ln f_{a,b,c,r}(x)]' \le 0, \quad x \in \mathbb{R}^+. \tag{18}$$

Combination of (16) and (18) leads directly to

$$r\left[x\ln(a+x) - x\ln x - \frac{a(x+b)}{x+a} - c + \psi(x+1) - \frac{\ln\Gamma(x+1)}{x}\right] \le 0, \quad x \in \mathbb{R}^+. \tag{19}$$

Since the limit of the left-hand side of (19) as $x \to 0+$ is $r[-b\,\mathrm{sgn}(a) - c]$, it follows clearly that

$$rc \ge -br\,\mathrm{sgn}(a). \tag{20}$$

On the other hand, utilization of Lemma 2 in (19) as $x \to \infty$ gives

$$r\left[x\ln\left(1+\frac{a}{x}\right) - \frac{a(x+b)}{x+a} - c + 1 - \frac{\ln(x+1)}{2x} + O\left(\frac{1}{x}\right)\right] \le 0. \tag{21}$$

Since the limit of the left-hand side of (21) as $x \to \infty$ is $r(1-c)$, it is deduced apparently that

$$rc \ge r. \tag{22}$$

Combining (20) and (22) yields

$$rc \ge \max\{r, -br\,\mathrm{sgn}(a)\}. \tag{23}$$

The proof of Theorem 1 is complete. □

Proof of Theorems 2 and 3. For $n \geq 2$, differentiating and induction yield

$$\begin{aligned}
[\ln f_{a,b,c,r}(x)]^{(n)} &= r\left\{\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{(-1)^k k!}{x^{k+1}}\psi^{(n-k-1)}(x+1)\right.\\
&\quad +(-1)^n(n-2)!\left[\frac{1}{(x+a)^{n-1}}-\frac{1}{x^{n-1}}\right]+(-1)^{n-1}(n-1)!\left[\frac{b-a}{(x+a)^n}-\frac{b}{x^n}\right]\\
&\quad \left.-\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{(-1)^k(k+1)!}{x^{k+2}}\psi^{(n-k-2)}(x+1)+(-1)^n(n-1)!\frac{c}{x^n}\right\}\\
&= r(-1)^n(n-2)!\left[\frac{(n-1)(b+c)-x}{x^n}+\frac{(n-1)(a-b)+(x+a)}{(x+a)^n}\right]+\frac{r\theta_n(x)}{x^{n+1}},
\end{aligned} \tag{24}$$

where

$$\theta_n(x)=\sum_{k=0}^{n}\frac{(-1)^{n-k}n!x^k\psi^{(k-1)}(x+1)}{k!} \tag{25}$$

with

$$\psi^{(-1)}(x+1)=\ln\Gamma(x+1) \quad\text{and}\quad \psi^{(0)}(x+1)=\psi(x+1).$$

It is easy to see that

$$\theta_n'(x)=x^n\psi^{(n)}(x+1). \tag{26}$$

Eq. (24) can be written as

$$(-1)^n x^{n+1}[\ln f_{a,b,c,r}(x)]^{(n)}+(-1)^{n+1}r\theta_n(x)=r(n-2)!\left\{(n-1)(b+c)x-x^2+\frac{x^{n+1}[(n-1)(a-b)+(x+a)]}{(x+a)^n}\right\}. \tag{27}$$

Let

$$g_n(x)=\frac{n(n-1)+(n-1)(b+c)x-2x^2}{x^{n+1}} \tag{28}$$

and

$$h_n(x)=\frac{an(a-b)(n-1)+(2a-b)(n-1)(x+a)+2(x+a)^2}{(x+a)^{n+1}}. \tag{29}$$

By induction and for $n \geq 2$, it is standard to show that

$$g_n'(x)=-(n-1)g_{n+1}(x) \quad\text{and}\quad g_2^{(n-2)}(x)=(-1)^n(n-2)!g_n(x), \tag{30}$$

$$h_n'(x)=-(n-1)h_{n+1}(x) \quad\text{and}\quad h_2^{(n-2)}(x)=(-1)^n(n-2)!h_n(x). \tag{31}$$

Direct computation along with (26) and formula (8) in Lemma 1 shows that

$$\begin{aligned}
\frac{\mathrm{d}\left\{(-1)^n x^{n+1}[\ln f_{a,b,c,r}(x)]^{(n)}\right\}}{\mathrm{d}x} &= (-1)^n r x^n\psi^{(n)}(x+1)+r(n-2)!\left\{(n-1)(b+c)-2x\right.\\
&\quad \left.+\frac{x^n[an(a-b)(n-1)+(2a-b)(n-1)(x+a)+2(x+a)^2]}{(x+a)^{n+1}}\right\}\\
&= rx^n\left\{(-1)^n\psi^{(n)}(x+1)+(n-2)!\left[\frac{(n-1)(b+c)-2x}{x^n}\right.\right.\\
&\quad \left.\left.+\frac{an(a-b)(n-1)+(2a-b)(n-1)(x+a)+2(x+a)^2}{(x+a)^{n+1}}\right]\right\}\\
&= rx^n\left\{(-1)^n\psi^{(n)}(x)+\frac{n!}{x^{n+1}}+(n-2)!\left[\frac{(n-1)(b+c)-2x}{x^n}\right.\right.\\
&\quad \left.\left.+\frac{an(a-b)(n-1)+(2a-b)(n-1)(x+a)+2(x+a)^2}{(x+a)^{n+1}}\right]\right\}\\
&= rx^n\left\{(-1)^n\psi^{(n)}(x)+(n-2)![g_n(x)+h_n(x)]\right\}.
\end{aligned} \tag{32}$$

Substituting (30) and (31) into (32) reveals that

$$\frac{\mathrm{d}\left\{(-1)^n x^{n+1}[\ln f_{a,b,c,r}(x)]^{(n)}\right\}}{\mathrm{d}x} = (-1)^n r x^n [\psi''(x) + g_2(x) + h_2(x)]^{(n-2)}. \tag{33}$$

It is easy to verify that

$$r[\psi''(x) + g_2(x) + h_2(x)] = \phi_{a,b,c,r}(x), \tag{34}$$

where $\phi_{a,b,c,r}(x)$ is defined by (12). Therefore, combining (13), (33) and (34) yields for $n \geq 2$ that

$$\frac{\mathrm{d}\{(-1)^n x^{n+1}[\ln f_{a,b,c,r}(x)]^{(n)}\}}{\mathrm{d}x} = x^n \int_0^\infty r\rho_{a,b,c}(t) t^{n-1} \mathrm{e}^{-xt} \mathrm{d}t, \quad x \in \mathbb{R}^+, \tag{35}$$

where $\rho_{a,b,c}(t)$ is defined by (14).

Standard argument shows that for $t \in \mathbb{R}^+$

$$0 < \frac{t}{\mathrm{e}^t - 1} < 1, \tag{36}$$

$$0 \leq \frac{1 - \mathrm{e}^{-at}}{t} \leq a, \tag{37}$$

and

$$\frac{a(b-a)t + b - 2a}{\mathrm{e}^{at}} \in \begin{cases} [b - 2a, 0], & \text{if } b \leq a; \\ \left(b - 2a, (b-a)\exp\left(\dfrac{a}{a-b}\right)\right], & \text{if } a < b \leq 2a; \\ \left(0, (b-a)\exp\left(\dfrac{a}{a-b}\right)\right], & \text{if } b > 2a. \end{cases} \tag{38}$$

Combining (14) and (36)–(38) gives for $t \in \mathbb{R}^+$

$$\rho_{a,b,c}(t) > \begin{cases} b - 2a - 1 + c, & \text{if } b \leq a; \\ b - 2a - 1 + c - (b-a)\exp\left(\dfrac{a}{a-b}\right), & \text{if } b > a \end{cases} \tag{39}$$

and

$$\rho_{a,b,c}(t) < \begin{cases} 2a + c, & \text{if } b \leq 2a; \\ b + c, & \text{if } b > 2a. \end{cases} \tag{40}$$

From (35), (39) and (40), it can be concluded that for $n \geq 2$ and $x \in \mathbb{R}^+$

$$\frac{\mathrm{d}\{(-1)^n x^{n+1}[\ln f_{a,b,c,r}(x)]^{(n)}\}}{\mathrm{d}x} > 0 \tag{41}$$

if one of the following conditions is satisfied:

(1) $r > 0$, $b \leq a$ and $c \geq 1 + 2a - b$;
(2) $r > 0$, $b > a$ and $c \geq 1 + 2a - b + (b-a)\exp(\frac{a}{a-b})$;
(3) $r < 0$, $b \leq 2a$ and $c \leq -2a$;
(4) $r < 0$, $b > 2a$ and $c \leq -b$.

From (25) and (27), it is easy to see that for $n \geq 2$

$$\lim_{x \to 0+} \left\{(-1)^n x^{n+1}[\ln f_{a,b,c,r}(x)]^{(n)}\right\} = 0. \tag{42}$$

Considering (41) and (42) gives for $n \geq 2$

$$(-1)^n [\ln f_{a,b,c,r}(x)]^{(n)} > 0, \quad x \in \mathbb{R}^+, \tag{43}$$

in particular,

$$[\ln f_{a,b,c,r}(x)]'' > 0, \quad x \in \mathbb{R}^+ \tag{44}$$

if one of the conditions (1)–(4) is satisfied.

By using Lemma 2 and from (16), it is obtained that

$$\lim_{x \to \infty} [\ln f_{a,b,c,r}(x)]' = 0. \tag{45}$$

Combining (44) and (45) yields

$$[\ln f_{a,b,c,r}(x)]' < 0, \quad x \in \mathbb{R}^+. \tag{46}$$

Consequently, inequality (43) holds for $n \in \mathbb{N}$ if any of the conditions (1)–(4) is satisfied. The proof of Theorems 2 and 3 is complete. □

Proof of Theorem 4. The sufficiency follows from conclusion (2) in Theorem 3. The necessity comes from (2) in Theorem 1. □

Acknowledgments

The second author was supported in part by the China Scholarship Council and the NSF of Henan University, China.

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